Segmentation *and classification* of a *Hawkes* process

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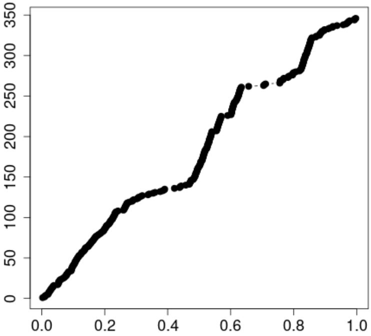
Problem

Counting process

Overnight recording of bat cries in continuous time



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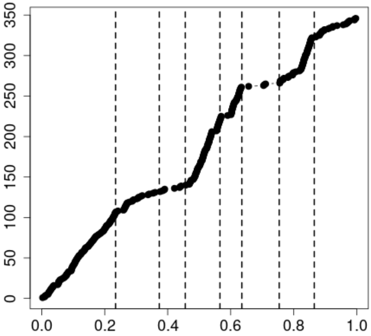
- Can we detect changes in the occurrence of events?



Problem

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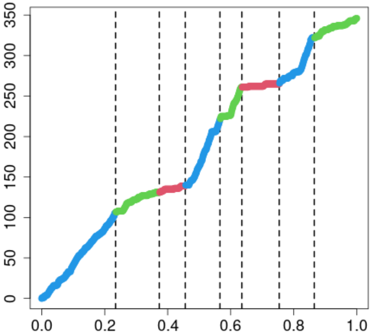
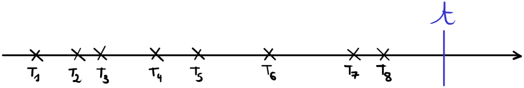
Overnight recording of bat cries in continuous time

- Can we detect changes in the occurrence of events?
- Can we associate each time period with some underlying behavior?



Point process

Reminder.



- $(T_k)_{k \geqslant 1}$ a random collection of points
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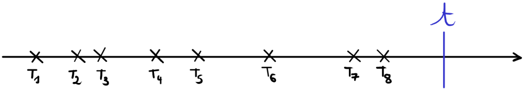

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Examples

- Homogeneous Poisson process: $\lambda(t) \equiv \lambda$
- Heterogeneous Poisson process: $\lambda(t) =$ deterministic function
- Hawkes process: $\lambda(t) =$ random function of the past

Segmentation (& classification) of a point process

Aim

1. Propose a set of reasonably realistic models;
2. Design an (efficient) algorithm to get the parameter estimates;
3. Choose among the models.

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Example Segmentation of a Poisson process [DBLR23]:

1. Model = Poisson process with piece-wise constant intensity function;
2. Algorithm = dynamic programming in (less than) $\mathcal{O}(N(T)^2)$;
3. Model selection = cross validation (using thining)

Outline

Segmentation of a Poisson process (1/3)

Model.

Change-points

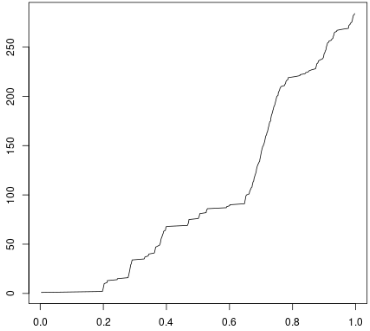
$$(\tau_0 =)0 < \tau_1 \cdots < \tau_{K-1} < 1(= \tau_K)$$

For $t \in I_k =]\tau_{k-1}, \tau_k]$:

$$\lambda(t) = \lambda_k$$

$\rightarrow$ Continuous piece-wise linear cumulated intensity function

Bat cries[a]



[a] source: Vigie-Chiro program, Y. Bas, CESCO-MNHN

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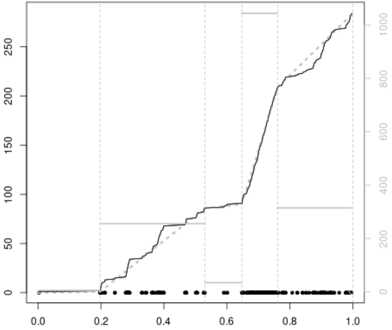

[a] source: Vigie-Chiro program, Y. Bas, CESCO-MNHN

- Segmentation algorithm: find the 'optimal' (τ, λ) in a reasonnable time
- Model selection: choose K

Segmentation of a Poisson process (2/3)

Classical contrasts (negative log-likelihood, least-square) are

- additive wrt the segments and
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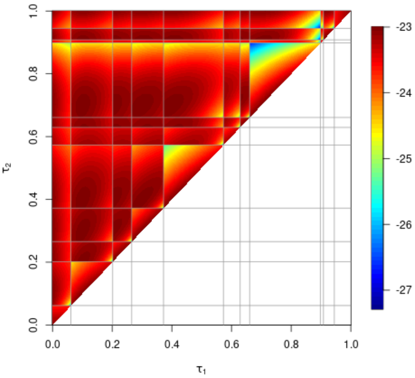

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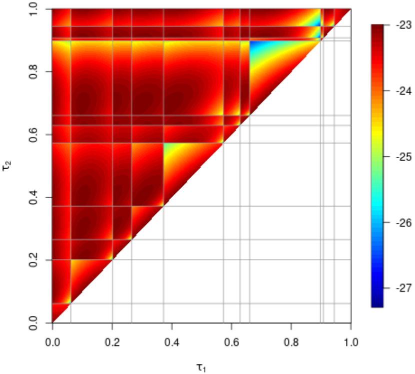
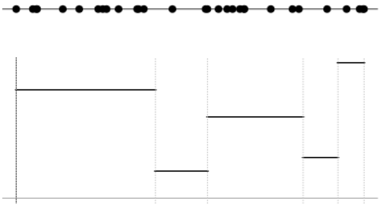


→ The set of optimal change points in included in

$$\{T_1^-, T_1, T_2^-, T_2, \dots T_i^-, T_i, \dots, T_n^-, T_n\}$$

→ The continuous optimization problem turns into a discrete optimization problem
→ Dynamic programming algorithm $= \mathcal{O}(n^2)$.

Segmentation of a Poisson process (3/3)

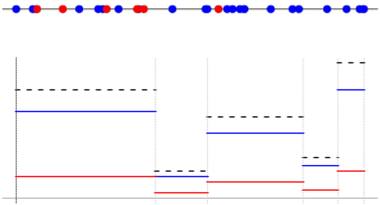


Lazy model selection. Thining property:

- independent processes with proportional internsities and common change point;
- cross-validation procedure to choose K.

[1] Modèles statistiques pour des données dépendantes et applications

Segmentation of a Poisson process (3/3)



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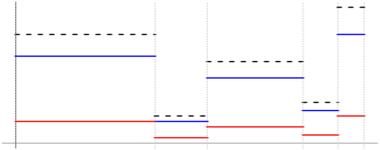


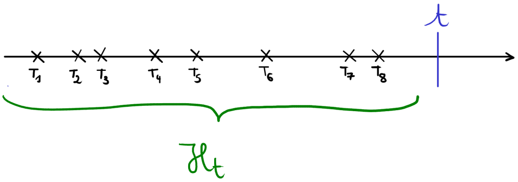
But segmenting a Poisson process

1. Does not provide any classification (although doable);
2. Does not account for the the self exciting (or inhibiting) nature of some processes;
3. Does not fit the scope of RMR2024[1].

[1] Modèles statistiques pour des données dépendantes et applications

Outline

Univariate Hawkes process



(Conditional) intensity function for the Hawkes process [Haw71]:

$$\lambda(t \mid \mathcal{H}_t) = \lambda(t) = \lambda_0 + \sum_{T_k < t} h(t - T_k)$$

- λ_0 = baseline
- h = kernel = influence of past events

Self-exciting exponential Hawkes process

$$\lambda(t) = \lambda_0 + \sum_{T_k < t} a e^{-b(t - T_k)}$$

Self exciting: Each event increases the probability of observing another event

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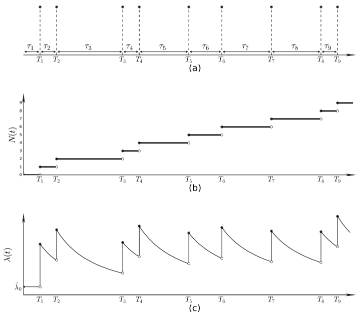

- Exponential kernel function $h(t) = ae^{-bt}$
- $a \geqslant 0$ to ensure that λ is non negative
- $a/b < 1$ to ensure stationarity
- Applications: sismology, epidemiology, neuroscience, ecology, ...

Discrete time Hawkes process

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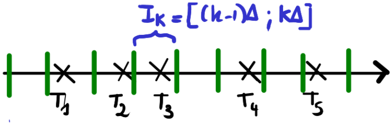
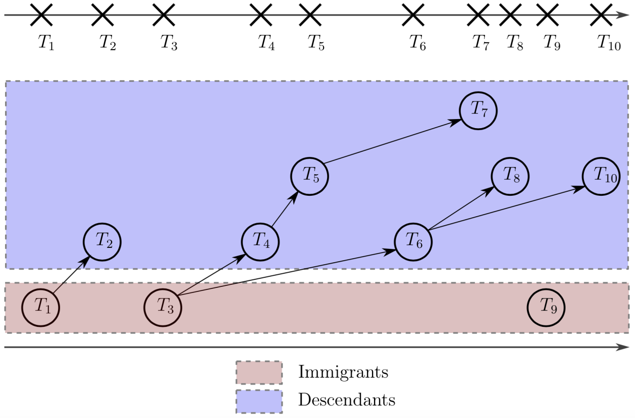
Discretization [Seo15]

- $I_k = [\tau_{k-1}; \tau_k]$ with $\tau_k = k\Delta$
- $N_k = N(I_k)$ the number of events on I_k



- Distribution of $(N_k)_{k \geqslant 1}$?

Cluster representation



- Immigrants arrive at rate λ_0
- Each immigrant or descendant produces new individuals at rate $h(t - T)$

Discrete time Hawkes process

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Count distribution

$$N_k \stackrel{\Delta}{=} B_k + \sum_{\ell \leqslant k-1} \sum_{T \in I_\ell} M_T(I_k) + R_k$$

- $B_k \sim \mathcal{P}(\lambda_0 \Delta)$ discrete immigrant process
- $M_T(I_k) \sim \mathcal{P}\left(c(a, b, \Delta) e^{-b(\tau_{k-1} - T)}\right)$ descendants of $T < \tau_{k-1}$
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Approximation when Δ is small

- $M_T(I_k) \simeq \mathcal{P}\left(c(a, b, \Delta) e^{-b(\tau_{k-1} - \tau_{\ell-1})}\right)$ for $T \in I_\ell = [\tau_{\ell-1}; \tau_\ell]$
- $R_k \simeq 0$

Markovian reformulation

Approximation of N_k

$$Y_k \mid \{Y_\ell\}_{\ell \leqslant k-1} \sim \mathcal{P}\left(\mu + \sum_{\ell=1}^{\infty} \alpha \beta^\ell Y_{k-\ell}\right),$$

with $\mu = \lambda_0 \Delta$ and α, β depending on a, b, Δ.

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$\{Y_k\}_{k\geqslant 1}$ is not a Markov chain but defining $\{U_k\}_{k\geqslant 1}$

$$U_1 = 0, \qquad U_k = \alpha Y_{k-1} + \beta U_{k-1},$$

so that

$$Y_k|(U_{k-1}, Y_{k-1}) \sim \mathcal{P}\left(\alpha Y_{k-1} + \beta U_{k-1}\right),$$

then we have that

$$\{(Y_k, U_k)\}_{k\geqslant 1}$$

is a Markov Chain.

Outline

Discrete time Hawkes HMM

Model: Q hidden states

- Hidden path: $\{Z_k\}_{k \geqslant 1}$ homogeneous Markov chain with transition matrix π
- Observed counts: for $k \geqslant 1$, set $U_1 = 0$ and

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Assumptions:

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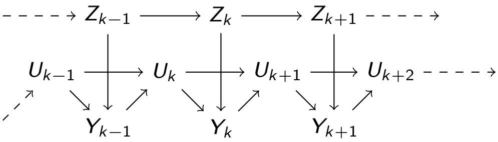
Graphical model:



Inference

Aim: Infer the parameter $\theta = ((\mu_q)_{1 \leqslant q \leqslant Q}, \pi)$

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EM algorithm: [DLR77]

$$\theta^{(h+1)} = \underbrace{\arg\max_\theta}_{\text{M step}} \underbrace{\mathbb{E}_{\theta^{(h)}}}_{\text{E step}} [\log p_\theta(Y, Z) \mid Y]$$

- E step: Evaluate $\ell^{(h)}(\theta) = \mathbb{E}_{\theta^{(h)}}[\log p_\theta(Y, Z) \mid Y]$ (forward-backward recursion)
- M step: Gradient descent, computing $\nabla_\theta \ell^{(h)}(\theta)$ by recursion

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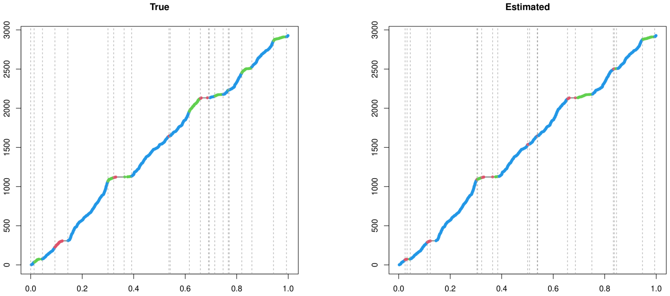
By-product: Classification

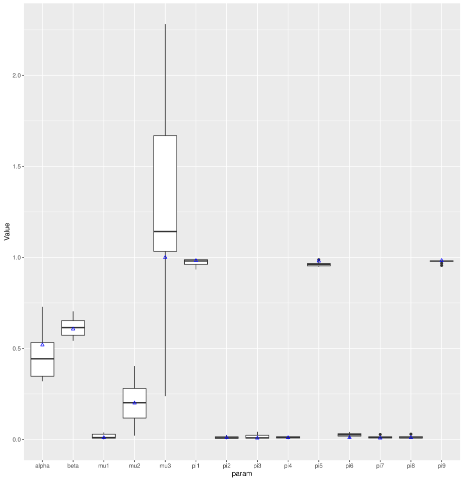
$$\widehat{Z}_k = \arg\max_q P_{\widehat{\theta}}\{Z_k = q \mid Y\}, \qquad \widehat{Z} = \arg\max_z p_{\widehat{\theta}}(Y, Z = z)$$

Synthetic data: classification

	States	1	2	3
Hawkes HMM	1	273.7	28.7	14.2
	2	37	166.3	96.9
	3	4.7	24.7	353.8
Poisson HMM	1	181	122.8	12.8
	2	136	111.1	53.1
	3	45.4	115.2	222.6



Synthetic data: parameter estimation



Outline

Model selection

Aim: select the number of hidden states Q

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Penalized likelihood:

$$\log p_\theta(Y) = \mathbb{E}_\theta[\log p_\theta(Y, Z) \mid Y] + \mathcal{H}(p_\theta(Z \mid Y))$$

→ Standard criterion for discrete time HMM

$$BIC(Q) = \log p_{\widehat{\theta}_Q}(Y) - pen(\widehat{\theta}_Q),$$
$$ICL(Q) = \log p_\theta(Y) - \mathcal{H}(p_{\widehat{\theta}_Q}(Z \mid Y)) - pen(\widehat{\theta}_Q)$$

with

$$pen(\widehat{\theta}_Q) = \frac{1}{2}\log(N)(Q^2 + 2)$$

where N = number of time steps (i.e. discretized intervals) = tuning parameter

Goodness-of-fit

Time-change theorem [DVJ03] A sequence $(T_k)_{k \geqslant 1}$ is a realization of N if and only if $(\Lambda(T_k))_{k \geqslant 1}$ is a realization of a homogeneous Poisson process with unit intensity. where

$$\Lambda(t) = \int_0^t \lambda(u) du \qquad \text{(Compensator)}$$

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Goodness-of-fit test

- H_0: "$(T_k)_{k \geqslant 1}$ is a realization of a HMM-Hawkes process with parameter θ".
- Kolmogorov-Smirnov test between

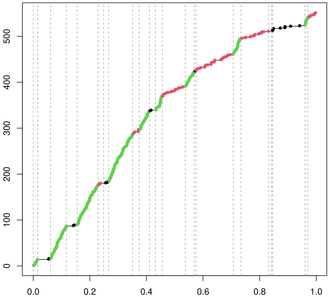
$$\left(\Lambda_{\widehat{\theta}}(T_{k+1}) - \Lambda_{\widehat{\theta}}(T_k)\right)_{k \geqslant 1}$$

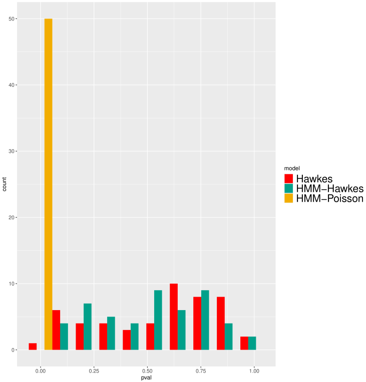
  and an exponential distribution $\mathcal{E}(1)$.

Comments

- Same test for alternative models (Hawkes, HMM-Poisson)
- Train/test samples (resampling procedure, [RBRGTM14])

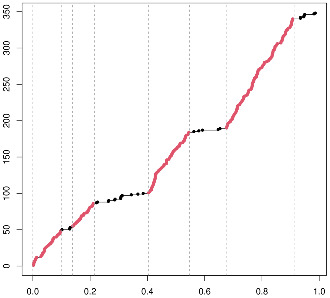
Synthetic data: Goodness-of-fit (1/2)

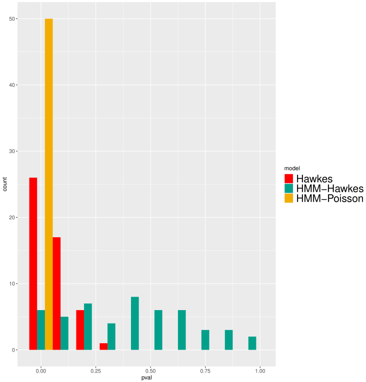




- The test rejects the homogeneous Poisson but does not differentiate the homogeneous Hawkes process from the HMM-Hawkes process.

Synthetic data: Goodness-of-fit (2/2)

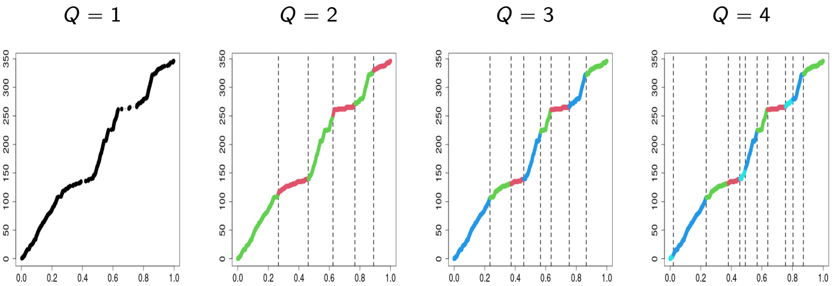




- The test is able to detect that the point process is neither an homogeneous Hawkes nor a HMM-Poisson process

Preliminary results on bat cries

Back to the recording of bat cries over one night:



- $Q = 2, 3$, hidden states? ($\widehat{Q}_{BIC} = 1$ or 2, depending on N)
- States = behavior (transit, foraging), species?

Outline

Some future works (1/2)

Multivariate Hawkes process: Consider p simultaneous processes $(N^{(i)})_{1 \leqslant i \leqslant p}$ (i.e. p neurons, bat species, ...)

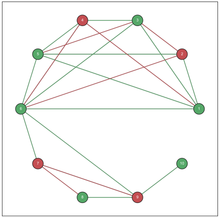
$$\lambda^{(i)}(t) = \lambda_0^i + \sum_{i=1}^{M} \sum_{T_k^j < t} h_{i,j}(t - T_k^j)$$

Exponential version:

$$h_{i,j}(t - T_k^j) = a_{i,j} e^{-b(t - T_k^j)}$$

where sparse interaction matrix $A = [a_{i,j}]_{1 \leqslant i,j \leqslant p}$

- Interaction network between neurons, species, ...



Some future works (2/2)

Modelling inhibition: Non-linear Hawkes process

$$\lambda(t) = \phi\left(\lambda_0 + \sum_{T_k \leqslant t} h(t - T_k)\right)$$

with $h < 0$.

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Effect of the hidden state: State-dependent parameters α and/or β

$$Y_k \mid \{Y_\ell\}_{\ell \leqslant k-1} \sim \mathcal{P}\left(\mu_{Z_k} + \sum_{\ell=1}^{\infty} \alpha_{Z_k} (\beta_{Z_k})^\ell Y_{k-\ell}\right)$$

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Backup: Discrete HMM

Conversion formulas from continuous to discrete Hawkes

$$\alpha = \frac{e^{b\Delta} - 1}{b}, \qquad \beta = e^{-b\Delta}$$

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3-step initialization

- Homogeneous Hawkes for the reproduction parameters α and β (`hawkesbow` R package [Che21])
- Poisson-HMM for the rates $\mu_1, \dots, \mu_Q$
- Correction $\mu_k \to \widetilde{\mu}_k$ to account for reproduction rate

Backup: GoF

Change-time for the recording of bat cries over one night:

$Q = 1$ $Q = 2$ $Q = 3$ $Q = 4$