Markov-switching (discrete-time) Hawkes process

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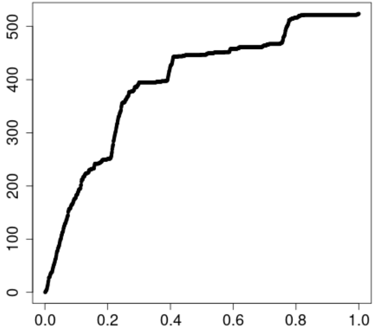
'Motivation'

Counting process

Overnight recording of bat cries in continuous time

'Motivation'



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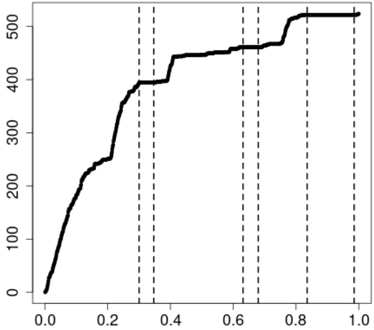
Overnight recording of bat cries in continuous time

- Can we detect changes in the distribution of events?



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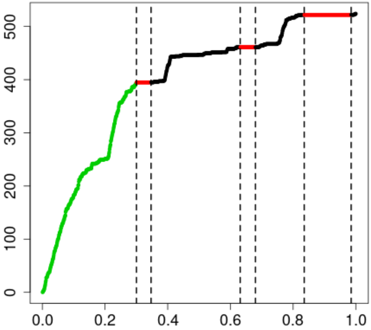
Overnight recording of bat cries in continuous time

- Can we detect changes in the distribution of events?
- Can we associate each time period with some underlying behavior?



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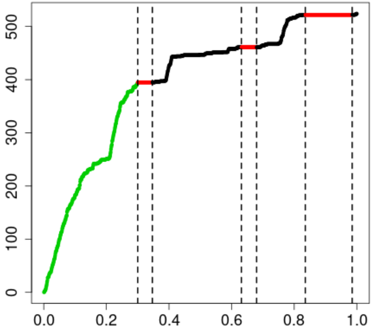
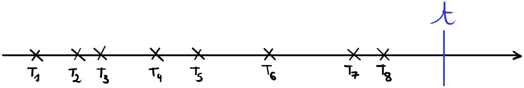


Modelling. Point process with (latent) Markov switching regime.

Point process

Point process

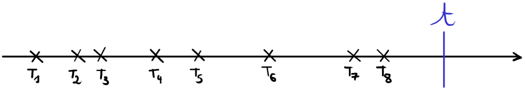
Reminder.



- $(T_k)_{k \geqslant 1}$ a random collection of points
- Count process $H(t) = \sum_{k \geqslant 1} \mathbb{I}\{T_k \leqslant t\}$
- Intensity function $\lambda(t)$: immediate probability of observing an event at time t

Point process

Reminder.



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Examples

- Homogeneous Poisson process: $\lambda(t) \equiv \lambda$
- Heterogeneous Poisson process: $\lambda(t) =$ deterministic function
- Hawkes process: $\lambda(t) =$ function of the past events $=$ random function

Outline

(Discrete) Hawkes process

- Continuous-time Hawkes process
- Discrete-time Hawkes process
- Markovian representation

Discrete Markov switching Hawkes process

- Model
- Identifiability & Inference

Simulation study

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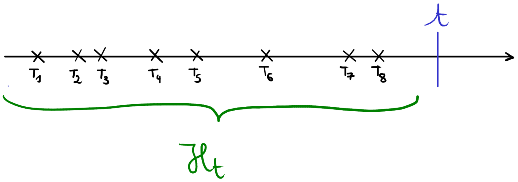
Discussion

Univariate Hawkes process



(Conditional) intensity function for the Hawkes process [Haw71]:

$$\lambda(t) = \lambda(t \mid \mathcal{H}_t) = \lambda_0 + \sum_{T_k < t} h(t - T_k)$$

- λ_0 = baseline
- h = kernel = influence of past events

Self-exciting exponential Hawkes process

$$\lambda(t) = \lambda_0 + \sum_{T_k < t} a e^{-b(t - T_k)}$$

Self exciting: Each event increases the probability of observing another event

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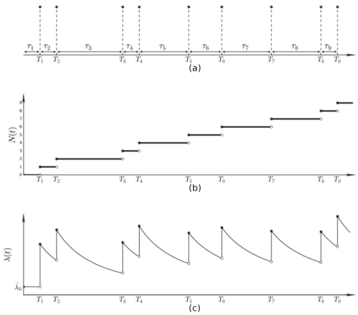
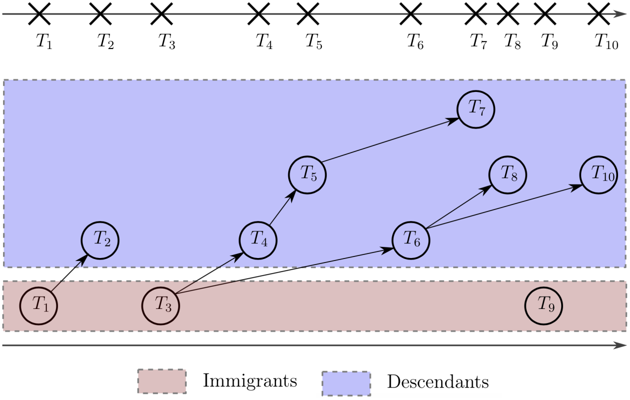


- Exponential kernel function $h(t) = ae^{-bt}$
- $a \geqslant 0$ to ensure that λ is non negative
- $a/b < 1$ to ensure stationarity
- Applications: sismology, epidemiology, vulcanology, neurosciences, ecology, ...

Cluster representation [HO74]



- Immigrants arrive at rate λ_0
- Each immigrant or descendant produces new individuals at rate $h(t - T)$

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Discrete-time Hawkes process

Continuous time exponential Hawkes process

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Discrete-time Hawkes process

Continuous time exponential Hawkes process

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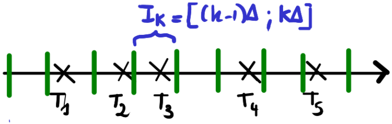
Discretization [Seo15,Kir16,Kir17]

- $I_k = [\tau_{k-1}; \tau_k]$ with $\tau_k = k\Delta$
- $H_k = H(I_k)$ the number of events on I_k



- Distribution of $(H_k)_{k \geqslant 1}$?

Decomposition of the count

H_k = number of events on $I_k = [\tau_{k-1}; \tau_k]$

$$H_k \stackrel{\Delta}{=} B_k + \sum_{\ell \leqslant k-1} \sum_{T \in I_\ell} M_T(I_k) + R_k$$

where

- B_k = number of immigrants within I_k:

$$B_k \sim \mathcal{P}(\mu)$$

  with $\mu = \lambda_0 \Delta$,

- $M_T(I_k)$ = number of descendants of $T < \tau_k$ within I_k:

$$M_T(I_k) \sim \mathcal{P}\left(\int_{I_k} a e^{-b(t-T)} \, \mathrm{d}t\right) = \mathcal{P}\left(\alpha e^{-b(\tau_k - T)}\right)$$

  with $\alpha = a(e^{b\Delta} - 1)/b$,

- R_k = number of descendants of points $T \in I_k$ within I_k

Discrete time Hawkes process

When Δ is small:

- $R_k \simeq 0$
- For $T \in I_\ell$: $e^{-b(\tau_k - T)} \simeq e^{-b(\tau_k - \tau_\ell)} = \beta^{k-\ell}$ with $\beta = e^{-b\Delta}$, so

$$\sum_{\ell \leqslant k-1} \sum_{T \in I_\ell} M_T(I_k) \overset{\Delta}{\simeq} \sum_{\ell \leqslant k-1} \sum_{T \in I_\ell} \mathcal{P}\left(\alpha \beta^{k-\ell}\right) \overset{\Delta}{=} \mathcal{P}\left(\sum_{\ell \leqslant k-1} H_{k-\ell} \alpha \beta^{\ell-1}\right)$$

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Discrete-time Hawkes process $Y = \{Y_k\}_{k \leqslant 1}$.

$$Y_k \mid (Y_\ell)_{\ell \leqslant k-1} \sim \mathcal{P}\left(\mu + \sum_{\ell=1}^{k-1} \alpha\beta^{\ell-1} Y_{k-\ell}\right)$$

See [Kir16] for the convergence toward a continuous-time Hawkes process.

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Markovian representation.

- Define

$$U_1 = 0, \qquad\qquad U_k = \sum_{\ell=1}^{k} \alpha\beta^{\ell-1} Y_{k-\ell},$$

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- we have for $k \geqslant 1$ (with $U_0 = Y_0 = 0$)

$$U_k = \alpha Y_{k-1} + \beta U_{k-1}, \qquad Y_k \mid U_k \sim \mathcal{P}(\mu + U_k).$$

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$\rightarrow$ $((Y_k, U_k))_{k \geqslant 1}$ forms a Markov Chain.

Graphical model

Discrete time Hawkes process.

$$(Y_k)_{k \geqslant 1} \sim \textit{Discrete Hawkes}(\mu, \alpha, \beta)$$

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Graphical model:

$$\begin{tikzcd}
& U_{k-1} \arrow[rr] \arrow[dr] & & U_k \arrow[rr] \arrow[dr] & & U_{k+1} \arrow[rr] \arrow[dr] & & U_{k+2} \arrow[r, dashed] & {} \\
{} \arrow[ur, dashed] & & Y_{k-1} \arrow[ur] & & Y_k \arrow[ur] & & Y_{k+1} \arrow[ur] & &
\end{tikzcd}$$

$(U_k)_{k \geqslant 1}$ = memory, $(Y_k)_{k \geqslant 1}$ = observed process.

$$\begin{aligned}
p\left(U_k, Y_k \mid (U_\ell, Y_\ell)_{\ell \leqslant k-1}\right) &= p\left(U_k, Y_k \mid U_{k-1}, Y_{k-1}\right) \\
&= p\left(U_k \mid U_{k-1}, Y_{k-1}\right) p\left(Y_k \mid U_k\right)
\end{aligned}$$

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Discrete time Hawkes HMM

Model: Q hidden states

- Hidden path: $(Z_k)_{k \geqslant 1}$ homogeneous Markov chain with Q states, transition matrix π and initial distribution ν:

$$(Z_k)_{k \geqslant 1} \sim MC_Q(\nu, \pi)$$

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$$(Y_k \mid (Y_\ell)_{\ell\leqslant k-1}, Z_k = q) \sim \mathcal{P}\left(\mu_q + \sum_{\ell=1}^{k-1} \alpha\beta^{\ell-1} Y_{k-\ell}\right)$$

or, for $k \geqslant 1$ (with $U_0 = Y_0 = 0$)

$$U_k = \alpha Y_{k-1} + \beta U_{k-1}, \qquad\qquad Y_k \mid U_k \sim \mathcal{P}\left(\mu_{Z_k} + U_k\right)$$

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Assumptions:

- The immigration rate μ varies with the hidden state
- The distribution of the number of offspring (α, β) does not vary with the hidden state

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$$
\begin{tikzcd}
{} \arrow[rr, dashed] & & Z_{k-1} \arrow[rr] \arrow[dd] & & Z_k \arrow[rr] \arrow[dd] & & Z_{k+1} \arrow[rr, dashed] \arrow[dd] & & {} \\
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$(Z_k)_{k\geqslant 1}$ = hidden path, $\quad (U_k)_{k\geqslant 1}$ = memory, $\quad (Y_k)_{k\geqslant 1}$ = observed process.

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$(Z_k)_{k\geqslant 1}$ = hidden path, $(U_k)_{k\geqslant 1}$ = memory, $(Y_k)_{k\geqslant 1}$ = observed process.

Remarks:

- The memory of the past is 'stored' in the variable U_k, which can still be computed recursively ($U_k = \alpha Y_{k-1} + \beta U_{k-1}$)
- The Markovian property still holds if the influence of the past varies with the hidden state ($\alpha \to \alpha_q, \beta \to \beta_q$).

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Identifiability

Proposition: The model parameter $\theta = (\nu, \pi, (\mu_q)_{1 \leqslant q \leqslant Q}, \alpha, \beta)$ is identifiable from the joint distribution $p_\theta^{Y_1, Y_2, Y_3}$:

$$\theta' \neq \theta \qquad \Rightarrow \qquad p_{\theta'}^{Y_1, Y_2, Y_3} \neq p_\theta^{Y_1, Y_2, Y_3}.$$

[1] The generic technique from [AMR09] does not apply here.

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Sketch of proof. Finite Poisson mixtures are identifiable [Tei61], so, because[1]

$$p_\theta^{Y_1, Y_2, Y_3}(x, y, z) = \sum_{1 \leqslant q, \ell, m \leqslant Q} \nu_q \pi_{q\ell} \pi_{\ell m} \mathcal{P}(x; \mu_q) \mathcal{P}(y; \mu_\ell + \alpha x) \mathcal{P}(z; \mu_m + \alpha\beta x + \alpha y),$$

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2. then α can be identified from $p_\theta(Y_2 \mid Y_1 = 1)$, [fix $x = 1$, sum over z]

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2. then α can be identified from $p_\theta(Y_2 \mid Y_1 = 1)$, [fix $x = 1$, sum over z]
3. then β can be identified from $p_\theta(Y_3 \mid Y_1 = 1, Y_2 = 0)$, [fix $x = 1$, $y = 0$]
4. then π can be identified from the joint mixture [sum over z]

$$p_\theta^{Y_1, Y_2}(x, y) = \sum_{1 \leqslant q, \ell \leqslant Q} \nu_q \pi_{q\ell} \mathcal{P}(x; \mu_q) \mathcal{P}(y; \mu_\ell + \alpha x),$$

which is proven identifiable.

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Inference

Aim: Infer the parameter θ

$$\widehat{\theta} = \arg\max_{\theta} \log p_{\theta}(Y)$$

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EM algorithm for HMM: [DLR77,CMR05]

$$\theta^{(h+1)} = \underbrace{\arg\max_{\theta}}_{\text{M step}} \underbrace{\mathbb{E}_{\theta^{(h)}}}_{\text{E step}} [\log p_\theta(Y, Z) \mid Y]$$

- E step: Evaluate $Q(\theta \mid \theta^{(h)}) = \mathbb{E}_{\theta^{(h)}}[\log p_\theta(Y, Z) \mid Y]$ (forward-backward recursion)
- M step: Gradient descent, computing $\nabla_\theta Q(\theta \mid \theta^{(h)})$ by recursion

Inference

Classification:

Marginal: $$\widehat{Z}_k = \arg\max_q P_{\widehat{\theta}}\{Z_k = q \mid Y\},$$

Joint (Viterbi): $$\widehat{Z} = \arg\max_z P_{\widehat{\theta}}\{Z = z \mid Y\}$$

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Model selection: Penalized likelihood

$$AIC_Q = \log p_{\widehat{\theta}_Q}(Y) - D_Q,$$

$$BIC_Q = \log p_{\widehat{\theta}_Q}(Y) - D_Q \frac{\log(N)}{2}$$

with D_Q = number of parameters $= 2 + Q^2$ and N = number of time bins.

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Simulation design ($Q = 3$)

- Baseline continuous parameters: $m^0 = [10,\ 200,\ 1000]$, $a^0 = 40$, $b = 160$

Simulation design ($Q = 3$)

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- Discretized process: $n = H(1)$

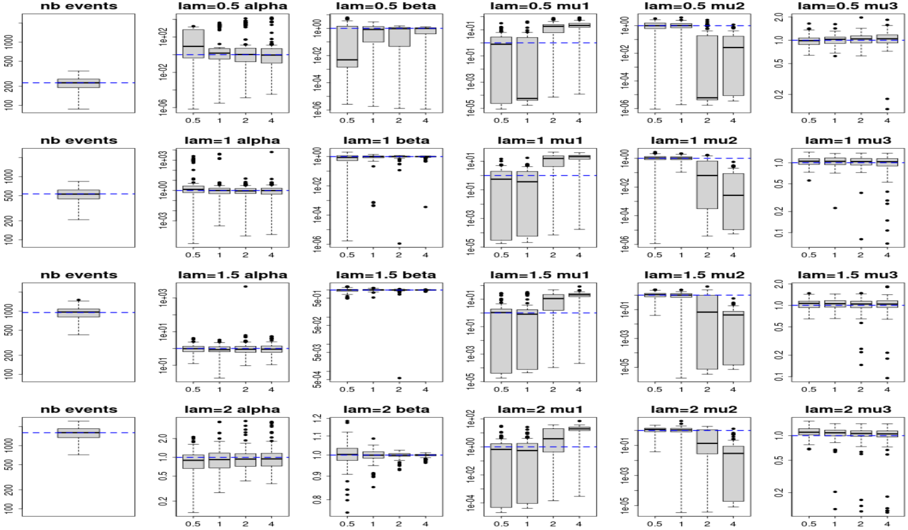
$$N = c\ n, \qquad c = 0.5,\ 1,\ 2,\ 4,$$

$$Y_k = H\left(\left[\frac{k-1}{N}; \frac{k}{N}\right]\right), \qquad k = 1, \dots N.$$

$\rightarrow$ not a discrete-time Hawkes process as defined earlier

Simulation results ($Q^* = 3$)

Parameter estimates. Distribution of $\widehat{\theta} - \theta^*$ ($Q = Q^*$)



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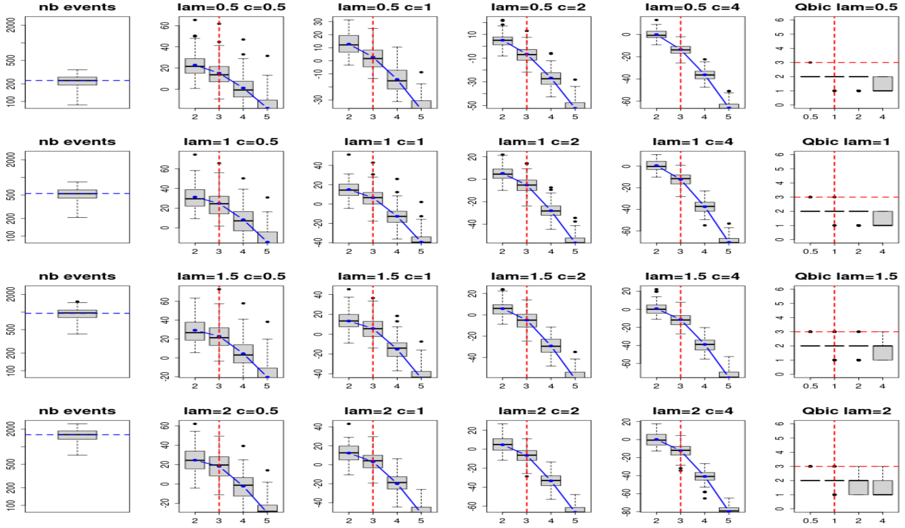
Model selection: BIC. Distribution of $BIC_Q - BIC_1$



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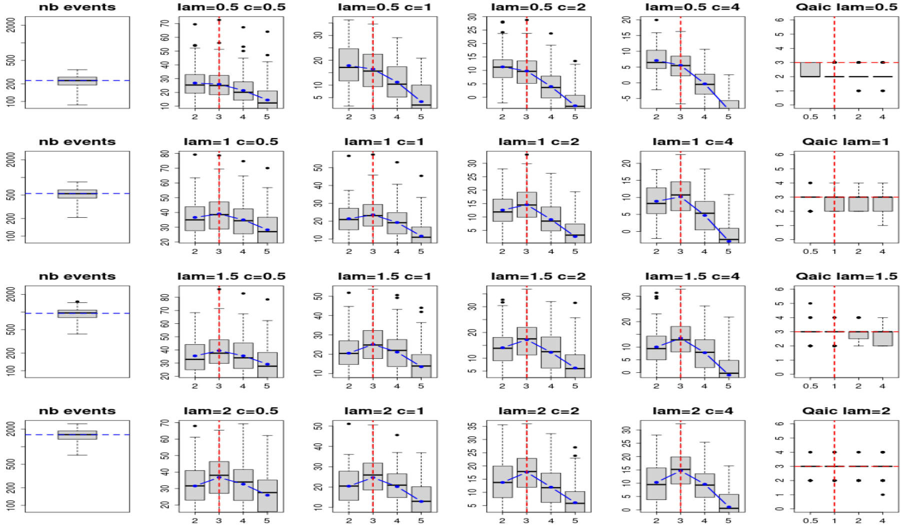

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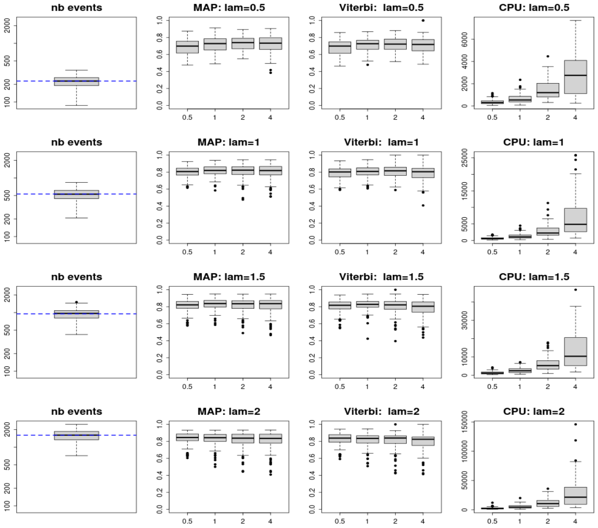

Simulation conclusions

- Inference easier when more signal (large λ)!!!
- Inference easier with thinner discretization step (large N)
  But at the price of a higher computational cost
- BIC does not capture the right number of states
  Sequences not simulated according to the model
- AIC does, with reasonable signal (λ) and discretization (N)
  Blind to the simulation shift from the model?

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  Blind to the simulation shift from the model?

Practical recommendations.

$$\text{Take } N = 2n \text{ and use AIC}$$

Outline

(Discrete) Hawkes process

- Continuous-time Hawkes process
- Discrete-time Hawkes process
- Markovian representation

Discrete Markov switching Hawkes process

- Model
- Identifiability & Inference

Simulation study

Illustrations

Discussion

Bat cries

Data set. 1555 overnight recordings all over France

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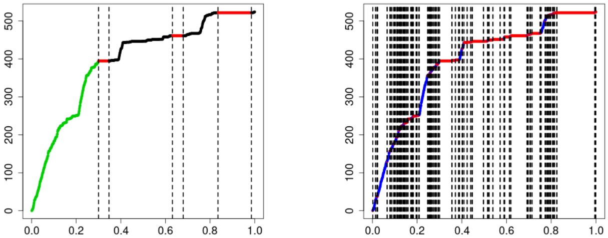
Poisson vs Hawkes / Homogeneous vs HMM. Best model based on AIC

	Poisson	Hawkes	Total
Homogeneous	34	353	387
Hidden Markov	24	1144	1168
Total	58	1497	1555

- Memory (95%) and heterogeneity (75%) are present in most sequences
- Hawkes-HMM best fits almost 3 sequences out of 4.

Example

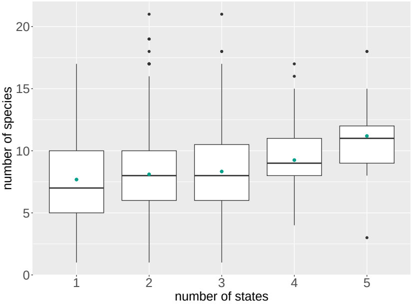
Hawkes HMM ($\widehat{Q} = 3$) Poisson HMM ($\widehat{Q} = 4$)



- Poisson-HMM needs many state changes to account for self-excitation
- Hawkes-HMM state changes do not correspond to slope changes

States and species

The number of bat species was also recorded



- The number of states does not match the number of species

Outline

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Summary

What we did.

- The discretized Hawkes process with exponential kernel is a Markov model

  ⇒ The discretized Markov switching Hawkes process with exponential kernel is a hidden Markov model

- The standard EM machinery applies to achieve maximum likelihood inference.

- Not shown: initialization based on existing estimation procedures for homogeneous Hawkes ([Che21],[CL22]) and Poisson HMM.

Discussion

What we did not do.

- Goodness-of-fit: 'Poissonisation' (on-going).
- Model selection: derive a proper (BIC?) criterion accounting for the discretization step.
- Understand the inferred latent states in terms of animal behavior, biogeography, species, ...

Discussion

What we did not do.

- Goodness-of-fit: 'Poissonisation' (on-going).
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- Understand the inferred latent states in terms of animal behavior, biogeography, species, ...

In parallel. With C. Dion-Blanc, D. Hawat and E. Lebarbier

- Efficient change-point detection ('segmentation') in (marked) Poisson & Hawkes processes.
  $\rightarrow$ Dynamic programming applies [DBHLR24].
- Segmentation-classification of Poisson processes.

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Discrete HMM

Conversion formulas from continuous to discrete Hawkes

$$\alpha = \frac{a(e^{b\Delta} - 1)}{b}, \qquad \beta = e^{-b\Delta}$$

Discrete HMM

Conversion formulas from continuous to discrete Hawkes

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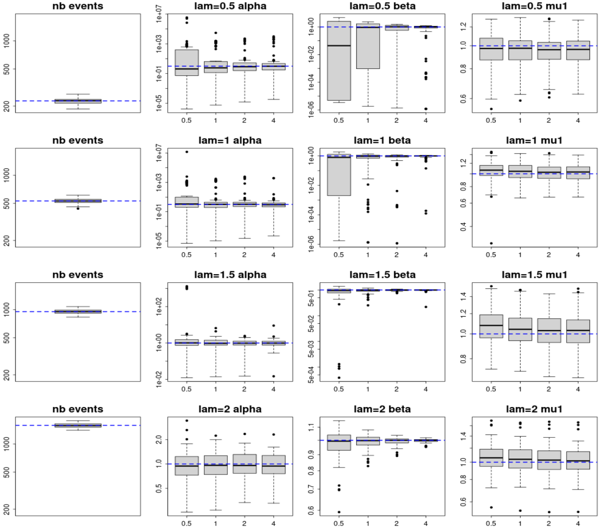
3-step initialization

- Homogeneous Hawkes for the reproduction parameters α and β (`hawkesbow` R package [Che21])
- Poisson-HMM for the rates $\mu_1, \dots, \mu_Q$ and transition π
- Correction $\mu_k \rightarrow \widetilde{\mu}_k$ to account for reproduction rate

Simulation results ($Q^* = 1$, $N = cn$, $m^0 = 400$)

Parameter estimates. Distribution of $\widehat{\theta} - \theta^*$ ($Q = Q^*$)



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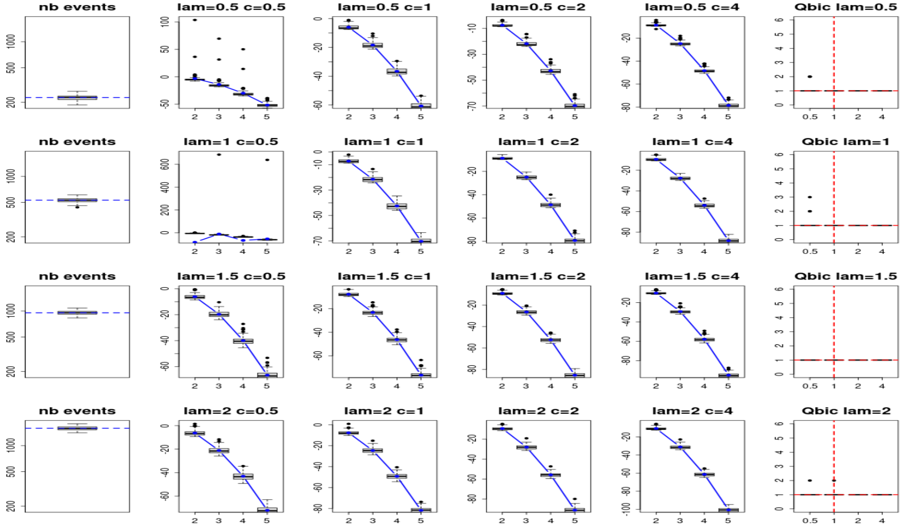

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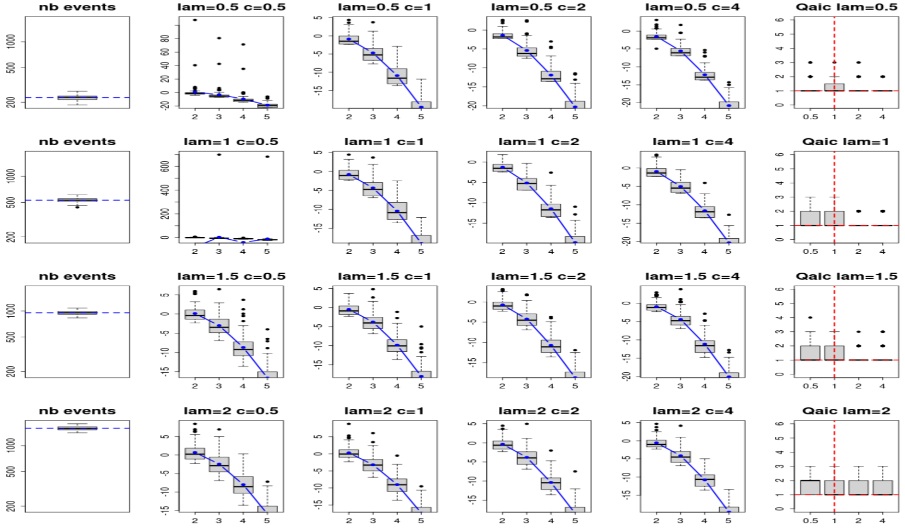
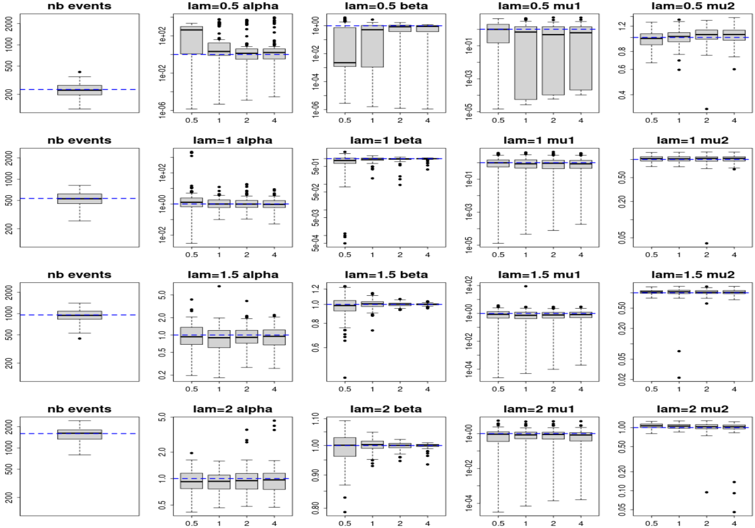


Simulation results ($Q^* = 2,\ N = cn,\ m^0 = [10,\ 800]$)

Parameter estimates. Distribution of $\widehat{\theta} - \theta^*$ ($Q = Q^*$)



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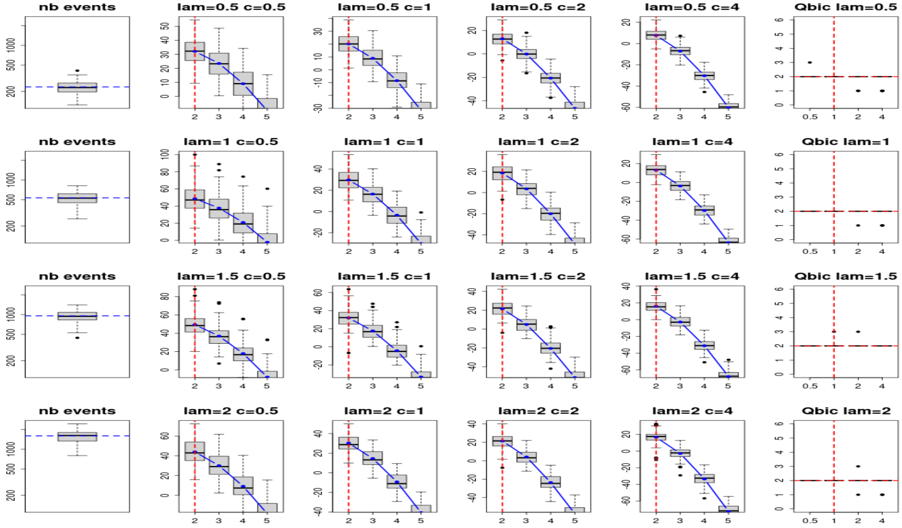

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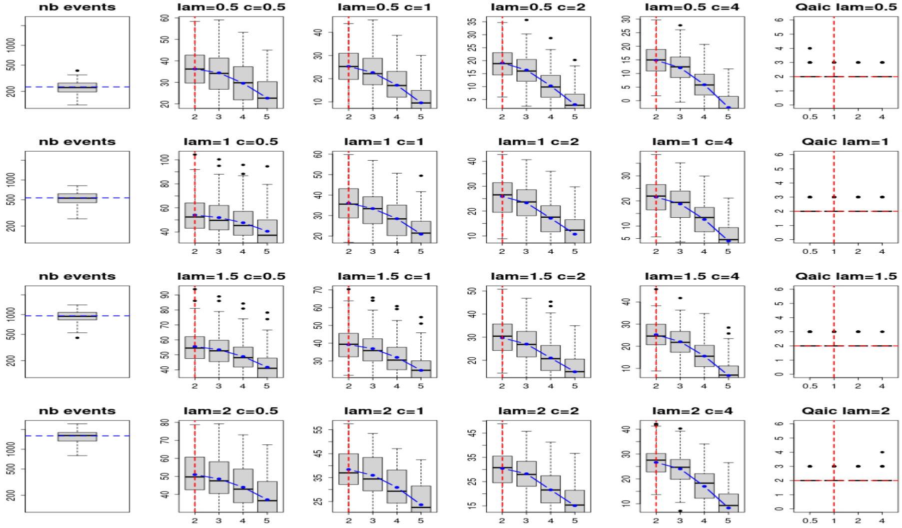

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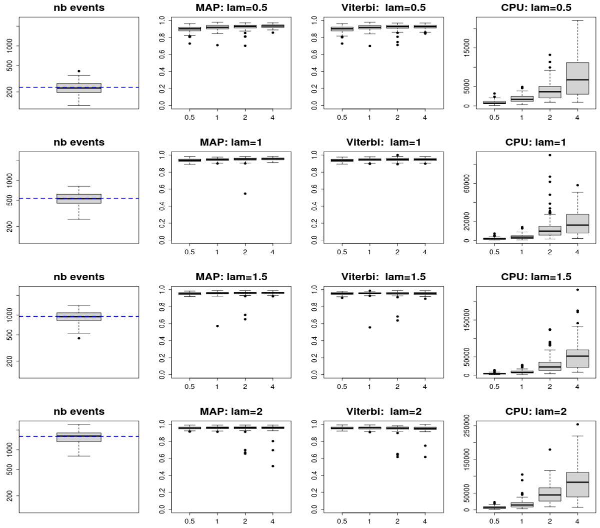

Model comparison for bat cries sequences

Poisson vs Hawkes / Homogeneous vs HMM. Best model based on BIC

	Poisson	Hawkes	Total
Homogeneous	132	775	907
Hidden Markov	21	627	648
Total	153	1402	1555

States and locations